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FINDING A NEW METHOD FOR THE SOLUTION OF NONLINEAR EQUATIONS $f(x) = 0$

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Abstract

In this paper, the weak points of some numerical methods which can be used to find the solutions of nonlinear equations $f(x) = 0$ are introduced.Then the new method (OhnmarNwe's method) is presented. And also the convergence of the new method is proved and comparison of the convergence for the new method and Newton's Method are expressed. Finally, the weak point of the new method is discussed.

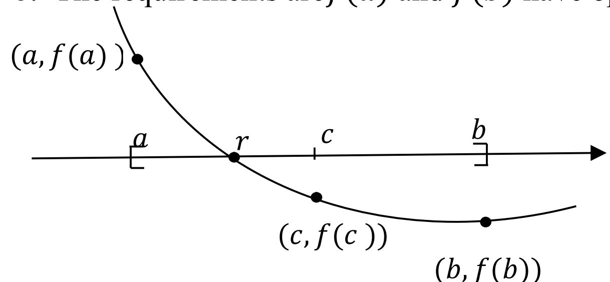
Introduction

We used the numerical methods for finding a zero of a continuous function. There are several methods, in which, we would like to discuss about the weak points of some numerical methods. We choose the methods are Bisection Method, Method of False Position, Secant Method, Newton – Raphson Method and Muller's Method. Our intention is to compare with the new method.

Weak Points of Some Methods

In this paper, we define the function f is continuous on the interval that we consider.

In **Bisection Method**, we need $f \in C[a, b]$ and to find $r \in [a, b]$ such that$f(r) = 0$. The requirements are$f(a)$ and $f(b)$ have opposite signs.



The formula is $c_n = \frac{a_n+b_n}{2}$for all n. Here only the average of the interval (i.e., $\frac{a_n+b_n}{2}$) is used.

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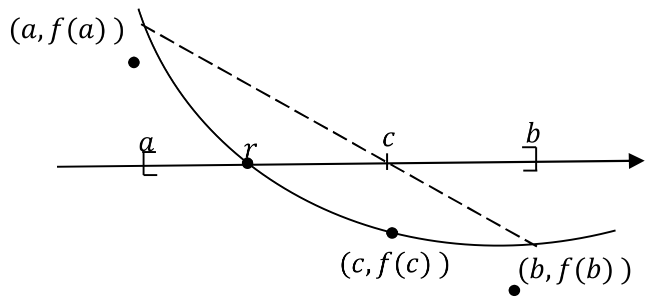
The convergence of the method is base on the sense that

$$|r - c_n| \leq \frac{b-a}{2^{n+1}} \text{for} n = 0,1, \dots \dots$$

The weakness of this method is the convergence speed is fairly slow. If $f(x) = 0$ has several roots in $[a, b]$, it is not easy to calculate as a different starting points and intervals must be used to find each root.

In **Method of False Position**, $f(a)$ and $f(b)$ need to have opposite signs. It is used the line joining the points $(a, f(a))$and $(b, f(b))$.



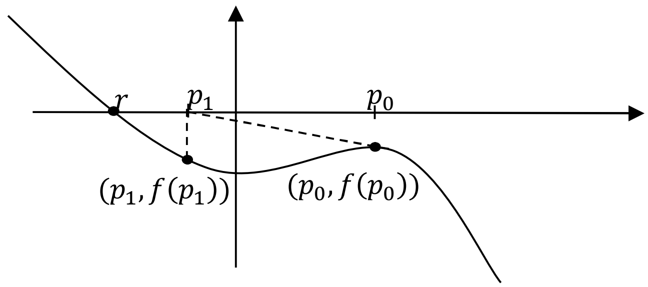
The formula is $c_n = b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$for all n.The size of the$f(b_n)$ and the interval $(b_n - a_n)$ are used.

The convergence of this method is based on the sense that $(b_n - a_n) \to 0$ as $n \to \infty$.

This method is faster than the bisection method. But it is also not easy to calculate for the several roots in an interval.

The weak points of the Bisection Method and Method of False Position are they need two initial points which have opposite signs of function value. Also these methods are not so easy to find the several roots.

In **Newton-Raphson Method (Newton's Method)**, $f(x), f'$(x) and f''(x) need to be continuous near a root. This method used the tangent lines.

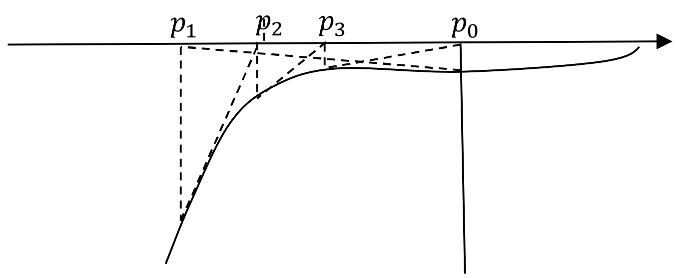


The formula is $p_k = \mathrm{p}_{\mathrm{k}-1} - \frac{\mathrm{f}(\mathrm{p}_{\mathrm{k}-1})}{\mathrm{f}'(\mathrm{p}_{\mathrm{k}-1})}$ for $k = 1,2, \ldots.$

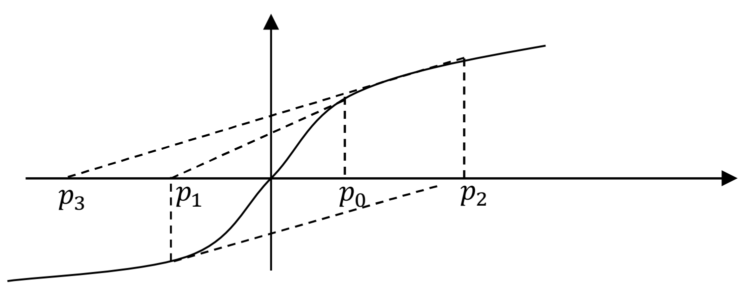
The convergence of this method is based on the Taylor polynomial and the fixed-point iteration.

This method is one of the most useful method. It is faster than the Bisection Method and Method of False Position. This method requires two function evaluations these are $f(p_{k-1})$and $f'(\mathrm{p}_{\mathrm{k}-1})$. And have difficulty if $f'(\mathrm{p}_{\mathrm{k}-1}) = 0$.

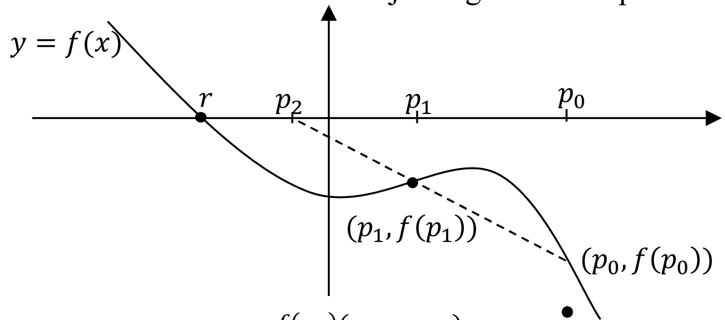
This means local maximum (or minimum) is in the interval. Sometimes the slope of $f'(p_0)$is small and the tangent line to the curve $y = f(x)$ is nearly horizontal. Then the sequence $\{p_k\}$ converges to some other root. Another possibility is cycling which occurs when the terms in the sequence $\{p_k\}$ tend to repeat.



Sometime the sequence does not converge for such a function $f(x) = \tan x$.

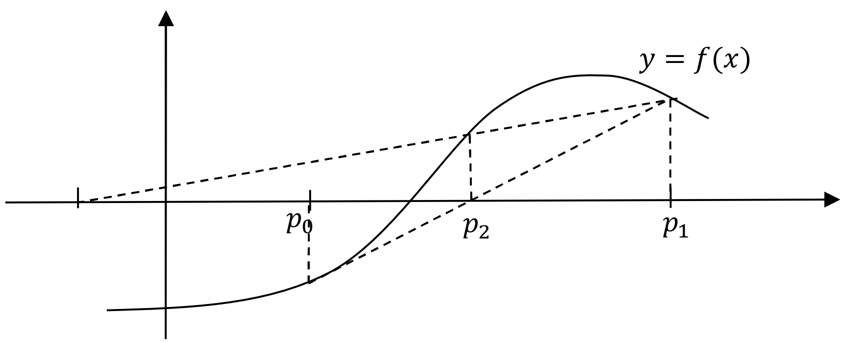


In **Secant Method**, two initial points $(p_0, f(p_0))$and $(p_1, f(p_1))$ near the root are needed. It is used the line joining these two points.

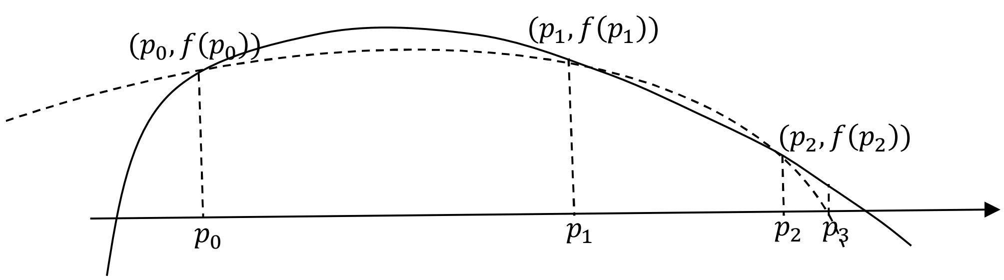


The formula is $p_{k+1} = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})}$for all k.

The convergence of this method is super linear. It is faster than a linear rate. It is almost as fast as Newton's Method. Secant Method needs only one function evaluation and is often faster in time, even though more iterates are needed to attain a similar accuracy with Newton's Method. The disadvantage of the secant method is sometime it may not converge when $f(p_k) \approx f(p_{k-1})$.



In **Muller's Method**, the three initial points $(p_0, f(p_0))$, $(p_1, f(p_1))$ and$(p_2, f(p_2))$ are needed. Theyare used to construct a parabola,second order polynomial, which is used to fit to the last three obtained points.



In this method, it is based on the variable $t = x - p_2$ and use the differences

$$h_0 = p_0 - p_2 \text{and} h_1 = p_1 - p_2.$$

And then use the quadratic polynomial, involving the variable,

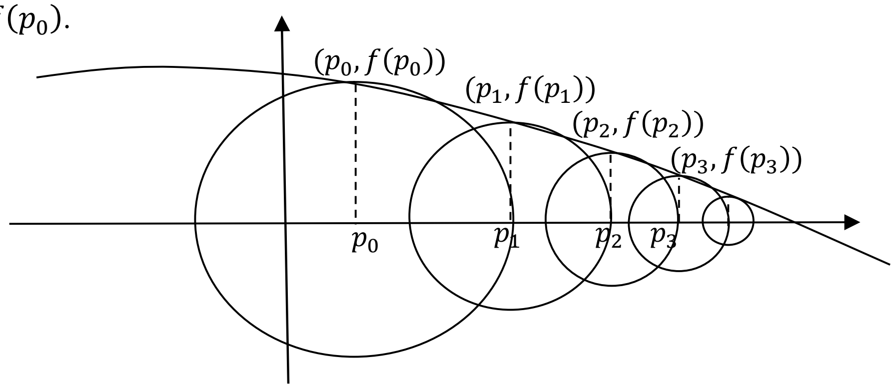
$$y = at^2 + bt + c\ .$$

This method can be used to find the imaginary roots and it is no needed to use derivatives. The convergence rate is faster than the Secant Method and almost as fast as Newton's Method.

The weakness of this method is three initial points are needed and extraneous roots can be found as this method used the quadratic formula.

New Method (Ohnmar Nwe's Method)

If $f(x)$ is continuous on the interval $[a, b]$ and it will across the X-axis at a root p that lie in the interval $[a, b]$. We draw a line by connecting the points $(p_0, 0)$and $(p_0, f(p_0))$. Then we construct a circle as centre $(p_0, 0)$ and radius $f(p_0)$.



Then this circle pass through the X-axis at $(p_1, 0)$. For the next step, construct the circle as centre $(p_1, 0)$ and radius $f(p_1)$. By proceeding, the centers of the circles are closer and closer to the root p. Here the equation of the circle with centre $(p_0, 0)$ and radius $f(p_0)$ is

$$(x - p_0)^2 + (y - 0)^2 = f(p_0)^2 \qquad (1)$$

This circle equation (1) pass through the X-axis at p_1. So that an equation relating p_1 and p_0 can be found.

$$(p_1 - p_0)^2 + (0 - 0)^2 = f(p_0)^2$$

$$p_1 = p_0 \mp f(p_0), \qquad (2)$$

When p_{k-1} and p_k are used in place of p_0 and p_1 the general rule is established as follow

$$p_k = p_{k-1} \mp f(p_{k-1}).$$

The convergence of this method is based on the idea that $|p_k - p_{k-1}| \to 0$ as $f(p_k) \to 0$.

The decision step for sign$(+)$ or $(-)$ is to analyze

$$|f(p_k)| < |f(p_{k-1})|.$$

Theorem (Ohnmar Nwe's Theorem)

Assume that $f \in C^1[a,b]$ and there exist a number $p \in [a,b]$ where $f(p) = 0$. Then there exist a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_k = g(p_{k-1}) = p_{k-1} \mp f(p_{k-1}) \text{ for } k = 1,2,3, \ldots$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$. (Here the function $g(x)$ is defined by $g(x) = x \mp f(x)$and it is used as the iteration function.)

Proof. We will use the fixed-point theorem to prove. We have to remind the fixed-point theorem.

(Fixed-point Theorem – Assume that $g(x)$ and $g'(x)$ are continuous on a balanced interval $(a,b) = (p - \delta, p + \delta)$ that contains the unique fixed point p and that the starting value p_0 is chosen in this interval.

If $|g'(x)| \leq K < 1$ for all $a \leq x \leq b$. Then the iteration $p_n = g(p_{n-1})$ will converge to p. In this case p is an attractive fixed-point.

If $|g'(x)| > 1$ for all $a \leq x \leq b$, then the iteration exhibits local divergence.)

In our method, the iteration function is

$$g(x) = x \mp f(x)$$

then $$g'(x) = 1 \mp f'(x)\ .$$

To be convergent, $g'(x)$ must be less than 1.

$$|g'(x)| = |1 \mp f'(x)| < 1$$

Therefore, a sufficient condition for p_0 to initialize a convergent sequence $\{p_k\}_{k=0}^{\infty}$ which converges to a root of $f(x) = 0$ is that $p_0 \in [p-\delta, p+\delta]$ and that δ be chosen so that

$$|1 \mp f'(x)| < 1 \quad \text{for all} \quad x \in [p-\delta, p+\delta].$$

Comparison of Newton's Method And Ohnmar Nwe's Method

Now we would like to express some examples that will show the comparison of convergent rate between Newton's method and our method.

Table 1. Comparison of convergent rate for the function $f(x) = x^3 + 3x + 2$ with $p_0 = 0$

k	Newton's Method	Ohnmar Nwe's Method
0	0	0
1	-0.6667	-2
2	-0.9333	
3	-0.9961	
4	-1.0000	

Table 2. Comparison of convergent rate for the function $f(x) = x^3 + 3x + 2$ with $p_0 = -2.5$

k	Newton's Method	Ohnmar Nwe's Method
0	-2.5	-2.5
1	-2.1250	-1.75
2	-2.0125	-1.9375
3	-2.0002	-1.9961
4	-2.0000	-2

Table 3. Comparison of convergent rate for the function $f(x) = x^3 + 3x + 2$ with $p_0 = -3$

k	Newton's Method	OhnmarNwe's Method
0	-3	-3
1	-2.3333	-1
2	-2.0667	
3	-2.0039	
4	-2.0000	

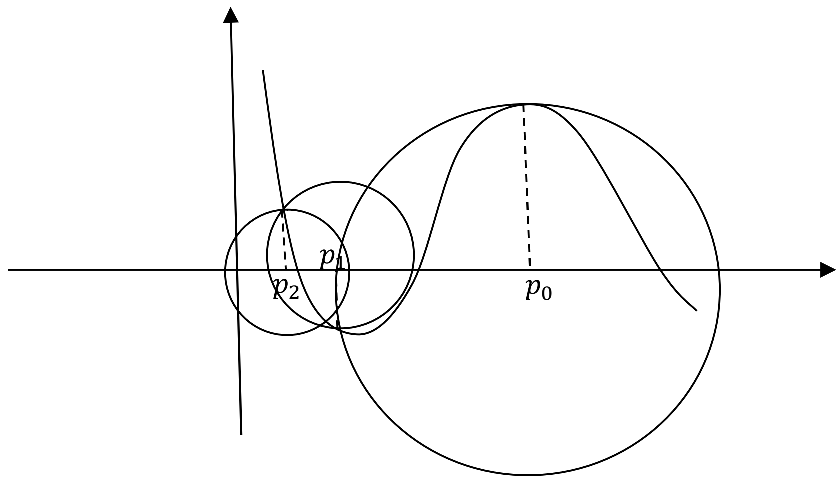
Table 4. Comparison of convergent rate for the function $f(x) = x^3 - 3x + 2$ with $p_0 = 0$

k	Newton's Method	OhnmarNwe's Method
0	0	0
1	0.6667	-2
2	0.8444	
3	0.9244	
4	0.9627	
5	0.9815	
6	0.9908	
7	0.9954	
8	0.9977	
9	0.9988	
10	0.9994	
11	0.9997	
12	0.9999	
13	0.9999	
14	1.0000	

Our method is needed only one initial point and convergent rate is faster than the Newton's method if the function has enough slope. The formula is also very simple and it is very easy to calculate.

Pitfall of Ohnmar Nwe's Method

There are some difficulties to use our method if the function $f(x)$ has several roots in the interval that we consider. At that condition, it may be jump of some roots although it gives a root . Another possibility is out of the interval if $f(p_0)$ is very large.



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